GREEN'S FUNCTIONS FOR BOUNDARY-VALUE PROBLEMS FOR THE HEAT EQUATION OVER REGIONS WITH UNIFORMLY MOVING BOUNDARIES

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Green's functions are obtained for a semi-infinite straight line with a uniformly moving boundary (10), (11), (12) and for a segment with boundaries moving uniformly and in parallel (16), (17), (18). For the solution a moving coordinate system is introduced and the method of Laplace transforms is applied.

It is well known that boundary-value problems for the heat equation over regions with moving boundaries lead to systems of Volterra integral equations of the second kind [1, 2]. In view of the difficulty associated with the solution of such systems, there have been introduced various artificial methods based on contour integration [3-5]. However, no general methods have been found. In the present work we shall derive Green's functions for the semi-infinite straight line with a uniformly moving boundary and for a straight-line segment with boundaries moving uniformly and in parallel, and we will thus settle the question of solving these two problems. In particular, one should note that Green's function for the first problem is obtained in closed form, and that for the second problem in a form analogous to well-known results for stationary boundaries.

Green's functions for the semi-infinite straight line. Let us introduce a system of coordinates which moves with the boundary according to $\chi(t) = vt$, so that $\xi = z - vt$ [4] Assume, also, that $\xi_0 = z_0 - vt_0$.

Under these assumptions the required Green's functions must satisfy the differential equation of heat conduction

$$a\frac{\partial^2 G}{\partial \xi^2} + v\frac{\partial G}{\partial \xi} - \frac{\partial G}{\partial t} = \frac{1}{2}\operatorname{sgn}(\xi_0 - \xi)\,\delta(\xi - \xi_0)\,\delta(t - t_0) \tag{1}$$

and the boundary conditions:

of the first kind

$$G(\xi,\ t;\ \xi_0,\ t_0)|_{\xi=0} = 0, \tag{2}$$

of the second kind

$$\frac{\partial}{\partial \xi} G(\xi,\ t;\ \xi_0,\ t_0)|_{\xi=0} = 0, \tag{3}$$

of the third kind

$$\left(\frac{\partial}{\partial \xi} - h\right) G(\xi,\ t;\ \xi_0,\ t_0)|_{\xi=0} = 0. \tag{4}$$

Furthermore, the condition

$$\lim_{\xi\to\infty} G(\xi,\ t;\ \xi_0,\ t_0) = 0 \tag{5}$$

must be satisfied.

Taking the Laplace transform (1) with respect to t, we obtain

$$a\frac{d^2\bar{G}}{d\xi^2} + v\frac{d\bar{G}}{d\xi} - s\bar{G} = \frac{1}{2}\operatorname{sgn}(\xi_0 - \xi)\,\delta(\xi - \xi_0)\exp(-st_0). \tag{6}$$

The general solution of (6) is of the form

$$\bar{G} = \left\{A - \frac{1}{2q}\left[\exp\left(-st_0 + \frac{v-q}{2a}\,\xi_0\right)\right]\right\}\left(\exp\frac{-v+q}{2a}\,\xi\right) +$$

$$+\left\{B+\frac{1}{2q}\left[\exp\left(-st_0+\frac{v+q}{2a}\xi_0\right)\right]\right\}\left(\exp\frac{-v-q}{2a}\xi\right) \quad (7)\ \text{(cont'd)}$$

for $\xi > \xi_0$, and of the form

$$\bar{G}=\left\{A+\frac{1}{2q}\left[\exp\left(-st_0+\frac{v-q}{2a}\xi_0\right)\right]\times\right.$$
$$\times\left(\exp\frac{-v+q}{2a}\xi\right)+\left\{B-\frac{1}{2q}\times\right.$$
$$\left.\times\left[\exp\left(-st_0+\frac{v+q}{2a}\xi_0\right)\right]\right\}\left(\exp\frac{-v-q}{2a}\xi\right) \quad (8)$$
$$(q=\sqrt{v^2+4as})$$

for $\xi < \xi_0$.

Boundary condition (5), applied to (7), requires that

$$A=\left[\exp\left(-st_0+\frac{v-q}{2a}\xi_0\right)\right]\Big/2q.$$

The constant B is found from the boundary condition at $\xi = 0$.

1. Boundary condition of the first kind (2). Equation (8) leads to the following expression for the constant B:

$$B_1=\frac{1}{2q}\left[\exp\left(-st_0+\frac{v+q}{2a}\xi_0\right)\right]-$$
$$-\frac{1}{q}\left[\exp\left(-st_0+\frac{v-q}{2a}\xi_0\right)\right].$$

Substituting the expressions for A and B into (7) and (8) and applying the inverse Laplace transformation, we obtain, after several simple transformations, the following expression for Green's function:

$$G(\xi,\ t;\ \xi_0,\ t_0)=\frac{1}{2\sqrt{\pi a(t-t_0)}}\exp\left\{-\frac{[v(t-t_0)+\xi-\xi_0]^2}{4a(t-t_0)}\right\}-$$
$$-\frac{1}{2\sqrt{\pi a(t-t_0)}}\exp\left\{\frac{v}{a}\xi_0-\frac{[v(t-t_0)+\xi+\xi_0]^2}{4a(t-t_0)}\right\}. \quad (9)$$

In a stationary system of coordinates this expression reduces to the form

$$G(z,\ t;\ z_0,\ t_0)=\frac{1}{2\sqrt{\pi a(t-t_0)}}\exp\left[-\frac{(z-z_0)^2}{4a(t-t_0)}\right]-$$
$$-\frac{1}{2\sqrt{\pi a(t-t_0)}}\exp\left[\frac{v}{a}(z_0-vt_0)-\frac{(z+z_0-2vt_0)^2}{4a(t-t_0)}\right]. \quad (10)$$

2. Boundary condition of the second kind (3). Differentiating (8) and equating the derivative at $\xi = 0$ with zero, we obtain the expression for the constant B:

$$B_2=\frac{1}{2q}\left[\exp\left(-st_0+\frac{v+q}{2a}\xi_0\right)\right]+$$
$$+\frac{1}{q}\left[\exp\left(-st_0+\frac{v-q}{2a}\xi_0\right)\right]-$$
$$-\frac{2v}{q(v+q)}\left[\exp\left(-st_0+\frac{v-q}{2a}\xi_0\right)\right].$$

After Laplace inversion, the solution to the problem is

$$G(\xi,\ t;\ \xi_0,\ t_0) = \frac{1}{2\sqrt{\pi a(t-t_0)}} \exp\left\{-\frac{[v(t-t_0)+\xi-\xi_0]^2}{4a(t-t_0)}\right\} +$$
$$+\frac{1}{2\sqrt{\pi a(t-t_0)}} \exp\left\{\frac{v}{a}\xi_0 - \frac{[v(t-t_0)+\xi+\xi_0]^2}{4a(t-t_0)}\right\} -$$
$$-\frac{v}{2a}\left(\exp\frac{v}{a}\xi_0\right) \operatorname{erfc}\left[\frac{v(t-t_0)+\xi+\xi_0}{2\sqrt{a(t-t_0)}}\right]. \tag{11}$$

3. Boundary condition of the third kind (4). By analogy with the preceding cases it can be shown that

$$B_3 = \frac{1}{2q}\left[\exp\left(-st_0 + \frac{v+q}{2a}\xi_0\right)\right] +$$
$$+\frac{1}{q}\left[\exp\left(-st_0 + \frac{v-q}{2a}\xi_0\right)\right] -$$
$$-\frac{4ah+2v}{q(2ah+v+q)} \exp\left(-st_0 + \frac{v-q}{2a}\xi_0\right)$$

and that after Laplace inversion the solution is

$$G(\xi,\ t;\ \xi_0,\ t_0) = \frac{1}{2\sqrt{\pi a(t-t_0)}} \exp\left\{-\frac{[v(t-t_0)+\xi-\xi_0]^2}{4a(t-t_0)}\right\} +$$
$$+\frac{1}{2\sqrt{\pi a(t-t_0)}} \exp\left\{\frac{v}{a}\xi_0 - \frac{[v(t-t_0)+\xi+\xi_0]^2}{4a(t-t_0)}\right\} - \tag{12}$$
$$-\frac{2ah+v}{2a} \exp\left[\frac{v}{a}\xi_0 + h(\xi+\xi_0) + h(ah+v)(t-t_0)\right] \times$$
$$\times \operatorname{erfc}\left[\frac{(2ah+v)(t-t_0)+\xi+\xi_0}{2\sqrt{a(t-t_0)}}\right].$$

In conclusion, note that one can easily derive from these formulas the well-known Green's functions for the semi-infinite straight line with a stationary boundary.

Green's functions for a straight segment with boundaries moving uniformly and in parallel. Taking into account the parallel motion of the boundaries, we shall use the same method of moving coordinates as in the preceding case. In the moving system of coordinates the problem reduces to solving the heat equation (1) under the (general) boundary conditions

$$\left(\alpha_1 - \beta_1\frac{\partial}{\partial\xi}\right) G(0,\ t;\ \xi_0,\ t_0) = 0, \tag{13}$$

$$\left(\alpha_2 + \beta_2\frac{\partial}{\partial\xi}\right) G(l,\ t;\ \xi_0,\ t_0) = 0. \tag{14}$$

As in the preceding case, the general solution in the transform space is of the form (7) for $\xi > \xi_0$, and (8) for $\xi < \xi_0$.

The constants A and B are determined by the boundary conditions at $\xi = 0$ and $\xi = l$, which form a system of two equation leading to

$$A(s) = \left[2\left(\alpha_1 - \beta_1\frac{-v-q}{2a}\right)\left(\alpha_2 + \beta_2\frac{-v-q}{2a}\right)\exp\left(-\frac{l-\xi_0}{2a}q\right) -\right.$$
$$-\left(\alpha_1 - \beta_1\frac{-v+q}{2a}\right)\left(\alpha_2 + \beta_2\frac{-v-q}{2a}\right)\exp\left(-\frac{l+\xi_0}{2a}q\right) -$$

$$-\left(\alpha_1-\beta_1\frac{-v-q}{2a}\right)\left(\alpha_2+\beta_2\frac{-v+q}{2a}\right)\times$$

$$\times\exp\frac{l-\xi_0}{2a}q\Big]\left[\exp\left(\frac{v}{2a}\xi_0-st_0\right)\right]\times$$

$$\times\Big\{2q\Big[\left(\alpha_1-\beta_1\frac{-v+q}{2a}\right)\left(\alpha_2+\beta_2\frac{-v-q}{2a}\right)\exp\left(-\frac{l}{2a}q\right)-$$

$$-\left(\alpha_1-\beta_1\frac{-v-q}{2a}\right)\left(\alpha_2+\beta_2\frac{-v+q}{2a}\right)\exp\frac{l}{2a}q\Big]\Big\}^{-1};$$

$$B(s)=\Big[2\left(\alpha_1-\beta_1\frac{-v+q}{2a}\right)\left(\alpha_2+\beta_2\frac{-v+q}{2a}\right)\times$$

$$\times\left(\exp\frac{l-\xi_0}{2q}\right)-\left(\alpha_1-\beta_1\frac{-v+q}{2a}\right)\times$$

$$\times\left(\alpha_2+\beta_2\frac{-v-q}{2a}\right)\exp\left(-\frac{l-\xi_0}{2q}\right)-$$

$$-\left(\alpha_1-\beta_1\frac{-v-q}{2a}\right)\left(\alpha_2+\beta_2\frac{-v+q}{2a}\right)\times$$

$$\times\exp\frac{l+\xi_0}{2a}q\Big]\left[\exp\left(\frac{v}{2a}\xi_0-st_0\right)\right]\times$$

$$\times\Big\{2q\Big[\left(\alpha_1-\beta_1\frac{-v+q}{2a}\right)\left(\alpha_2+\beta_2\frac{-v-q}{2a}\right)\exp\left(-\frac{l}{2a}q\right)-$$

$$-\left(\alpha_1-\beta_1\frac{-v-q}{2a}\right)\left(\alpha_2+\beta_2\frac{-v+q}{2a}\right)\exp\frac{l}{2a}q\Big]\Big\}^{-1}.$$

Substituting A and B into (7) and (8) we obtain Green's function in the transform space in the form of a ratio of two functions. Inverting the transform we use the expansion theorem. The transform of Green's function is a single-valued function of s with a denumerable set of simple poles lying on the negative real axis. Thus it is convenient to denote $(q/2a)\,i$ by γ, whereupon the denominator of the transform becomes

$$4a\gamma\,[(\alpha_1\alpha_2+\alpha_2\beta_1\omega-\alpha_1\beta_2\omega-\beta_1\beta_2\omega^2-\beta_1\beta_2\gamma^2)\sin\gamma l+$$
$$+(\alpha_1\beta_2+\alpha_2\beta_1)\gamma\cos\gamma l]\quad(\omega=v/2a),$$

and the numerator becomes (for $\xi>\xi_0$)

$$2\,[(\alpha_1\alpha_2+\alpha_2\beta_1\omega-\alpha_1\beta_2\omega-\beta_1\beta_2\omega^2+\beta_1\beta_2\gamma^2)\cos\gamma(l-\xi_0-\xi)+$$
$$+(\alpha_2\beta_1-\alpha_1\beta_2-2\beta_1\beta_2\omega)\gamma\sin\gamma(l-\xi_0-\xi)+$$
$$+(\alpha_2\beta_1+\alpha_1\beta_2)\gamma\sin\gamma(l+\xi_0-\xi)-$$
$$-(\alpha_1\alpha_2+\alpha_2\beta_1\omega-\alpha_1\beta_2\omega-\beta_1\beta_2\omega^2-\beta_1\beta_2\gamma^2)\cos\gamma(l+\xi_0-\xi)]\times$$
$$\times\exp(\omega(\xi_0-\xi)-st_0).$$

Henceforth our discussion of the problem shall depend on the type of boundary condition under consideration.

1. Boundary conditions of the first kind at both ends of the segment: $G(0,\ t;\ \xi_0,\ t_0)=G(l,\ t;\ \xi_0,\ t_0)=0$.

In this case $\beta_1=\beta_2=0$, and

$$G(\xi,\ t;\ \xi_0,\ t_0)=\frac{2}{l}\left[\exp\frac{v}{2a}(\xi_0-\xi)\right]\sum_{n=1}^{\infty}\sin\frac{\pi n\,\xi_0}{l}\times\qquad(15)$$

$$\times \sin\frac{\pi n \xi}{l} \exp\left[-\left(\frac{n^2\pi^2}{l^2} a + \frac{v^2}{4a}\right)(t - t_0)\right]. \qquad (15)$$

(cont'd)

In a stationary system of coordinates this solution can be written in the form

$$G(z, t; z_0, t_0) = \frac{2}{l}\left[\exp\frac{v}{2a}(z_0 - z)\right] \times$$

$$\times \sum_{n=1}^{\infty} \sin\frac{\pi n (z_0 - vt_0)}{l} \sin\frac{\pi n (z - vt)}{l} \times$$

$$\times \exp\left[-\left(\frac{n^2\pi^2}{l^2} a - \frac{v^2}{4a}\right)(t - t_0)\right]. \qquad (16)$$

2. Boundary conditions of the second kind at both ends of the segment: $\frac{\partial}{\partial \xi} G(0, t; \xi_0, t_0) = \frac{\partial}{\partial \xi} G(l, t; \xi_0, t_0) = 0$.

In this case $\alpha_1 = \alpha_2 = 0$, and

$$G(\xi, t; \xi_0, t_0) = \frac{2}{l}[\exp\omega(\xi_0 - \xi)] \times$$

$$\times \sum_{n=1}^{\infty} \frac{1}{\omega^2 + \pi^2 n^2/l^2}\left[\omega^2 \sin\frac{\pi n \xi_0}{l} \sin\frac{\pi n \xi}{l} + \right.$$

$$+ \frac{n^2\pi^2}{l^2} \cos\frac{\pi n \xi_0}{l} \cos\frac{\pi n \xi}{l} +$$

$$\left. + \frac{\omega\pi n}{l} \sin\frac{\pi n(\xi_0 + \xi)}{l}\right] \exp\left[-a\left(\frac{n^2\pi^2}{l} + \omega^2\right)(t - t_0)\right] \quad (\omega \neq 0). \qquad (17)$$

3. Boundary conditions of the third kind at both ends of the segment: $\left(h_1 - \frac{\partial}{\partial \xi}\right) G(0, t; \xi_0, t_0) = \left(h_2 + \frac{\partial}{\partial \xi}\right) \cdot G(l, t; \xi_0, t_0) = 0$.

Here $h_1 = \alpha_1/\beta_1$, $h_2 = \alpha_2/\beta_2$, and

$$G(\xi, t; \xi_0, t_0) = 2\exp\omega(\xi_0 - \xi) \times$$

$$\times \sum_{\gamma}^{\infty} \gamma[(h_1 + h_2)(\gamma l - \sin 2\gamma l/2) + 2\gamma \sin^2 \gamma l]^{-1} \times$$

$$\times \left\{(h_1 + h_2)\sin\gamma(l - \xi)\sin\gamma(l - \xi_0) + \sin\gamma l[\gamma\cos\gamma(l - \xi - \xi_0) - \right.$$

$$\left. - (h_1 + \omega)\sin\gamma(l - \xi_0 - \xi)]\right\} \exp[-a(\gamma^2 + \omega^2)(t - t_0)], \qquad (18)$$

where the summation extends over all positive roots of the characteristic equation

$$(h_1 h_2 + h_2\omega - h_1\omega - \omega^2 - \gamma^2)\sin\gamma l + (h_1 + h_2)\gamma\cos\gamma l = 0.$$

Green's functions for other boundary conditions can be obtained in an analogous manner.

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